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Differential properties for a class of Sobolev orthogonal polynomials ☆

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Abstract

We study the orthogonal polynomials with respect to a Sobolev inner product of the following type:

$$\langle f,g\rangle_s=\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}|B_h(\mathrm{e}^{\mathrm{i}\theta})|^2\,\frac{\mathrm{d}\theta}{2\pi}+\frac{1}{\lambda}\int_0^{2\pi} f'(\mathrm{e}^{\mathrm{i}\theta})\overline{g'(\mathrm{e}^{\mathrm{i}\theta})}\,\frac{\mathrm{d}\theta}{2\pi},\quad z=\mathrm{e}^{\mathrm{i}\theta},$$

where $B_h(z)$ is a complex polynomial of degree h, $\mathrm{d}\theta/2\pi$ is the normalized Lebesgue measure and λ is a positive real number.

The asymptotic behavior in the complex plane, as well as the differential equations satisfied by the orthogonal polynomials are obtained. As an application, two differential problems are solved, one of them is like a Dirichlet boundary value problem.

MSC: 33C47; 42C05

Keywords: Orthogonal polynomials; Minimal norm; Recurrence relation; Asymptotics; Differential equation

1. Introduction

The aim of this paper is to study asymptotic properties and differential properties of the sequence of monic polynomials orthogonal with respect to the following Sobolev inner product:

$$\langle f,g\rangle_s=\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}|B_h(\mathrm{e}^{\mathrm{i}\theta})|^2\,\frac{\mathrm{d}\theta}{2\pi}+\frac{1}{\lambda}\int_0^{2\pi} f'(\mathrm{e}^{\mathrm{i}\theta})\overline{g'(\mathrm{e}^{\mathrm{i}\theta})}\,\frac{\mathrm{d}\theta}{2\pi},\quad z=\mathrm{e}^{\mathrm{i}\theta},\tag{1}$$

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where $d\theta/2\pi$ is the normalized Lebesgue measure and $B_h(z)$ is a complex polynomial of degree h such that $B_h(0) \neq 0$, and λ is a positive real number.

In [2] we have studied the polynomials orthogonal with respect to inner products like (1), when the degree of the polynomial B_h is 1, obtaining some algebraic properties and the asymptotic behavior. Now we generalize these results and we obtain new properties.

One of the reasons for studying the Sobolev orthogonality is because of its connections with linear differential operators and spectral methods for boundary value problems. In the present work we construct a differential operator which is closely connected to the Sobolev inner product and we obtain the differential equations satisfied by the monic Sobolev orthogonal polynomials, as well as we solve a certain type of differential equation related with the operator.

The organization of the paper is as follows. In Section 2 we obtain an algebraic recurrence relation satisfied by the sequence of monic Sobolev orthogonal polynomials. In Section 3 we obtain suitable bounds for the coefficients of the polynomials and we study the asymptotic behavior of the monic Sobolev orthogonal polynomials in the complex plane. Finally, in the last section, the differential equations satisfied by the sequence of the monic Sobolev orthogonal polynomials are obtained. Moreover, as an application, two differential problems are solved, one of them is like a Dirichlet boundary value problem.

2. Algebraic properties of the monic Sobolev orthogonal polynomials

The Gram matrix of the inner product (1), with respect to the basis $\{z^n\}$ of the space of polynomials $\mathbb{P}$ is a $(2h+1)$-diagonal matrix. Indeed, if we denote by c_n the moments for the measure $d\mu(\theta) = |B_h(e^{i\theta})|^2 d\theta/2\pi$, we have

$$c_n = \langle z^n, 1\rangle_\mu = \int_0^{2\pi} e^{in\theta} |B_h(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

and therefore $c_n = 0$ if $n > h$ and $c_h \neq 0$. Thus, for $n \geqslant h+1$ we get $\langle z^n, z^k\rangle_s = 0$ for $k = 0, \dots, n-h-1$.

We denote by $\{\Phi_n\}$ the monic orthogonal polynomial sequence with respect to (1), which we shall write in a simpler form

$$\langle f, g\rangle_s = \langle f, g\rangle_\mu + \frac{1}{\lambda}\langle f', g'\rangle_\theta$$

and for simplicity we assume that μ is a probability measure, that is, $c_0 = 1$.

Theorem 1. *Let $\{\Phi_n\}$ be the monic orthogonal polynomial sequence with respect to* (1). *Then there exist h sequences $\{\alpha_{n,n-k}\}_{n \geqslant h+1}$, $(k = 1, \dots, h)$, such that*

$$z^n = \Phi_n(z) + \sum_{k=1}^{h} \alpha_{n,n-k} \Phi_{n-k}(z), \quad \textit{for } n \geqslant h+1, \tag{2}$$

where

$$\lim_{n\to\infty} \alpha_{n,n-k} = 0 \quad \textit{for } k = 1, \dots, h.$$

Moreover,

$$\alpha_{n,\,n-h} = \frac{c_h}{\|\Phi_{n-h}\|_s^2} \neq 0 \tag{3}$$

and there exists $N \in \mathbb{N}$ such that the sequence $\{|\alpha_{n,\,n-h}|\}_{n \geqslant N}$ is monotonically decreasing.

Proof. Let $n \geqslant h+1$. We can write

$$z^n = \Phi_n(z) + \sum_{k=1}^{n} \alpha_{n,\,n-k}\Phi_{n-k}(z),$$

with

$$\alpha_{n,\,n-k} = \frac{\langle z^n, \Phi_{n-k}\rangle_s}{\|\Phi_{n-k}\|_s^2}.$$

Since $\langle z^n, z^k\rangle_s = 0$ for $k = 0,\dots,n-h-1$, then $\langle z^n, \Phi_k\rangle_s = 0$ for $k = 0,\dots,n-h-1$, and therefore $\alpha_{n,\,n-k} = 0$ for $k = h+1,\dots,n$, and we obtain (2).

For $k = h$, it is immediate that

$$\alpha_{n,\,n-h} = \frac{\langle z^n, z^{n-h} + \cdots\rangle_s}{\|\Phi_{n-h}\|_s^2} = \frac{c_h}{\|\Phi_{n-h}\|_s^2}.$$

Taking into account the extremal property of the norms of the monic Sobolev orthogonal polynomials we get (see [1])

$$1 + \frac{n^2}{\lambda} = \|z^n\|_s^2 \geqslant \|\Phi_n\|_s^2 \geqslant \frac{n^2}{\lambda} \tag{4}$$

and this implies

$$\lim_{n\to\infty} \frac{\|\Phi_n\|_s^2}{n^2} = \frac{1}{\lambda} \tag{5}$$

and also

$$\exists N \in \mathbb{N} \text{ such that} \{\|\Phi_n\|_s^2\}_{n \geqslant N} \text{ is an increasing sequence.} \tag{6}$$

Then, from (3) and (6) we get that $\{|\alpha_{n,\,n-h}|\}_{n \geqslant N}$ is decreasing.

Finally, by computing the Sobolev norm in (2) and applying (4) we obtain

$$1 + \frac{n^2}{\lambda} = \|z^n\|_s^2 = \|\Phi_n\|_s^2 + \sum_{k=1}^{h} |\alpha_{n,\,n-k}|^2 \|\Phi_{n-k}\|_s^2 \geqslant \frac{n^2}{\lambda} + \sum_{k=1}^{h} |\alpha_{n,\,n-k}|^2 \|\Phi_{n-k}\|_s^2.$$

Therefore, for each $k = 1,\dots,h$

$$1 \geqslant |\alpha_{n,\,n-k}|^2 \|\Phi_{n-k}\|_s^2 \geqslant |\alpha_{n,\,n-k}|^2 \frac{(n-k)^2}{\lambda},$$

which implies $|\alpha_{n,\,n-k}|^2 \leqslant \lambda(n-k)^{-2}$ and thus $\lim_{n\to\infty} \alpha_{n,\,n-k} = 0$. $\square$

Corollary 1. (i) *The sequence* $\{\Phi_n\}$ *satisfies the following* $(h+2)$*-term forward recurrence relation*:

$$\Phi_{n+1}(z) = (z - \alpha_{n+1,n})\Phi_n(z) + \sum_{k=1}^{h-1}(\alpha_{n,\,n-k}z - \alpha_{n+1,n-k})\Phi_{n-k}(z) + \alpha_{n,\,n-h}z\Phi_{n-h}(z) \quad \textit{for } n \geqslant h+1. \tag{7}$$

(ii) *The sequence* $\{\Phi_n\}$ *satisfies the following* $(h+2)$*-term backward recurrence relation*:

$$\Phi_{n-h}(z) = \frac{1}{\alpha_{n,\,n-h}z}\Big[(-z + \alpha_{n+1,n})\Phi_n(z) - \sum_{k=1}^{h-1}(\alpha_{n,\,n-k}z - \alpha_{n+1,n-k})\Phi_{n-k}(z) + \Phi_{n+1}(z)\Big] \quad \textit{for } n \geqslant h+1. \tag{8}$$

3. Asymptotics and zeros

First we obtain bounds for the coefficients of the polynomials Φ_n that we need for deducing asymptotics.

Theorem 2. *Let* $\Phi_n(z) = z^n + \sum_{k=0}^{n-1} A_{n,\,k}z^k$ *and let* δ *such that* $0 < \delta < 1$. *Then there exist* $M > 0$ *and* $N \in \mathbb{N}$ *such that for* $n \geqslant N$

$$|A_{n,k}| \leqslant \delta^{n-k-h}\frac{M}{k^2} \quad (k \geqslant 1) \tag{9}$$

and

$$|A_{n,0}| \leqslant \delta^{n-h}M. \tag{10}$$

Proof. From the recurrence relation (2) we obtain the following linear homogeneous difference equation:

$$0 = A_{n,k} + \alpha_{n,n-1}A_{n-1,k} + \cdots + \alpha_{n,n-h}A_{n-h,k} \quad (k = 0, \ldots, n-h-1),$$

which can be written as

$$\begin{pmatrix} A_{n-h+1,k} \\ \vdots \\ A_{n,k} \end{pmatrix} = \mathbf{A}_n \begin{pmatrix} A_{n-h,k} \\ \vdots \\ A_{n-1,k} \end{pmatrix} \tag{11}$$

with

$$\mathbf{A}_n = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_{n,n-h} & -\alpha_{n,\,n-h+1} & \cdots & -\alpha_{n,\,n-2} & -\alpha_{n,\,n-1} \end{pmatrix}.$$

Since $\lim_{n\to\infty} \alpha_{n,n-i} = 0$ $(i = 1,\dots,h)$, then the sequence of matrices $\{\mathbf{A}_n\}$ converges to the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with spectral radius $\rho(\mathbf{A}) = 0$.

Since $\rho(\mathbf{A})=\inf\{\|\mathbf{A}\|$: with $\|\,\|$ induced by a vector norm$\}$; if we take δ such that $0 < \delta < 1$, then there exists a norm in $\mathbb{C}^h$ such that the corresponding matrix norm satisfies $\rho(\mathbf{A}) < \|\mathbf{A}\| < \delta$.

The convergence of $\mathbf{A}_n$ to $\mathbf{A}$ implies that for n large enough $\|\mathbf{A}_n\| < \delta$. For simplicity, and without loss of generality we may assume that $\|\mathbf{A}_n\| < \delta$, $\forall n$. Hence, if we take $k < n-h$ and apply (11) recursively we deduce

$$\begin{pmatrix} A_{n-h+1,k} \\ \vdots \\ A_{n,k} \end{pmatrix} = \mathbf{A}_n \begin{pmatrix} A_{n-h,k} \\ \vdots \\ A_{n-1,k} \end{pmatrix} = \mathbf{A}_n\mathbf{A}_{n-1}\dots\mathbf{A}_{k+h+1} \begin{pmatrix} A_{k+1,k} \\ \vdots \\ A_{k+h,k} \end{pmatrix}$$

and therefore

$$\left\| \begin{pmatrix} A_{n-h+1,k} \\ \vdots \\ A_{n,k} \end{pmatrix} \right\| \leqslant \delta^{n-k-h} \left\| \begin{pmatrix} A_{k+1,k} \\ \vdots \\ A_{k+h,k} \end{pmatrix} \right\|.$$

If we use that our norm is equivalent to $\|\,\|_\infty$ in $\mathbb{C}^h$, and take into account the following fact (see [1]):

$$|A_{n,k}| \leqslant \frac{\lambda}{k^2} \quad (k = 1,\dots,n-1), \tag{12}$$

then we deduce

$$|A_{n,k}| \leqslant \delta^{n-k-h}\,\frac{C\lambda}{k^2} \quad (k \geqslant 1) \quad \text{with } C > 0,$$

that is

$$|A_{n,k}| \leqslant \delta^{n-k-h}\,\frac{M}{k^2} \quad (k \geqslant 1).$$

Similarly, we can obtain that $|A_{n,0}| \leqslant \delta^{n-h}M$. $\square$

Since the Sobolev inner product is dominated by the Lebesgue measure involving the derivatives, the Sobolev orthogonal polynomials asymptotically resemble the orthogonal polynomials z^n corresponding to Lebesgue measure on the unit circle. Essentially, this is what we prove in the next theorem.

Theorem 3. *Let* $\delta > 0$. *Then*

$$\lim_{n\to\infty} \frac{\Phi_n}{z^n} = 1 \text{ uniformly for } |z| > \delta.$$

Proof. Let z be such that $0<\delta<|z|\leqslant 1$. Let $S=(M/\delta^h)\sum_{n=0}^{\infty}(\delta/|z|)^n$ with M defined as in Theorem 2.

For a fixed $\varepsilon>0$, let us take $N\in\mathbb{N}$ such that $1/N^2<\varepsilon/3S$. Also we take m such that $M/\delta^h(\delta/|z|)^m<\varepsilon/3N$ and p such that $\lambda h(|z|^h(p-h)^2)<\varepsilon/3$.

Then, for $n\geqslant\max\{N+h+1,m+N-1,p\}$ and n large enough such that (9) and (10) hold, we can write

$$\left|\frac{\Phi_n(z)}{z^n}-1\right|=\left|\sum_{k=0}^{n-1}\frac{A_{n,k}}{z^{n-k}}\right|\leqslant\sum_{k=n-h}^{n-1}\frac{|A_{n,k}|}{|z|^{n-k}}+\sum_{k=N}^{n-h-1}\frac{|A_{n,k}|}{|z|^{n-k}}+\sum_{k=0}^{N-1}\frac{|A_{n,k}|}{|z|^{n-k}}$$

and obtain appropriate bounds for each term. Applying (12)

$$\sum_{k=n-h}^{n-1}\frac{|A_{n,k}|}{|z|^{n-k}}<\sum_{k=n-h}^{n-1}\frac{\lambda}{|z|^{n-k}k^2}<\frac{\lambda h}{|z|^h(n-h)^2}<\frac{\varepsilon}{3}.$$

From (9) and (10) we obtain

$$\sum_{k=N}^{n-h-1}\frac{|A_{n,k}|}{|z|^{n-k}}<\sum_{k=N}^{n-h-1}\frac{\delta^{n-k-h}M}{|z|^{n-k}k^2}=\sum_{k=N}^{n-h-1}\frac{M}{\delta^h}\left(\frac{\delta}{|z|}\right)^{n-k}\frac{1}{k^2}$$
$$<\frac{\varepsilon}{3S}\sum_{k=N}^{n-h-1}\frac{M}{\delta^h}\left(\frac{\delta}{|z|}\right)^{n-k}<\frac{\varepsilon}{3}$$

and

$$\sum_{k=0}^{N-1}\frac{|A_{n,k}|}{|z|^{n-k}}=\sum_{k=1}^{N-1}\frac{|A_{n,k}|}{|z|^{n-k}}+\frac{|A_{n,0}|}{|z|^n}\leqslant\sum_{k=1}^{N-1}\frac{\delta^{n-k-h}M}{|z|^{n-k}k^2}+\frac{\delta^{n-h}M}{|z|^n}$$
$$\leqslant\sum_{k=0}^{N-1}\frac{M}{\delta^h}\left(\frac{\delta}{|z|}\right)^{n-k}<N\frac{\varepsilon}{3N}=\frac{\varepsilon}{3}.$$

Therefore for $\varepsilon>0$ there exists k such that

$$\forall n\geqslant k,\quad\left|\frac{\Phi_n(z)}{z^n}-1\right|<\varepsilon,\quad\forall z,\ \delta<|z|.$$

For $|z|>1$, the result can be obtained in a similar way . For this last case see [1]. □

Corollary 2. *Let $\delta>0$. For n large enough the zeros of Φ_n are in $|z|<\delta$.*

Proof. First of all, we note that the zeros of Φ_n are in a disk $|z|<1+K$, with $K>0$ because the sequences of coefficients are bounded. Then, from Theorem 3 and using Hurwitz's Theorem, we obtain the result. □

4. Differential properties and applications

Let F be the Carathéodory function associated with the measure μ, $F(z) = 1 + 2\sum_{k=1}^{h} \overline{c_k} z^k$ and let $G(z) = \frac{1}{2}(F(z) + \bar{F}(1/z)) = \sum_{k=-h}^{h} c_k z^{-k}$. If we denote by $D_r = \{z\colon |z| < r\}$ and the unit disk by $D = D_1$, it is well known that $F \in H(D)$ and $\Re F(z) > 0\ \forall z \in D$, and $\Re F(e^{i\theta}) = \mu'(\theta)$ a.e. in $[0, 2\pi)$ (see [4,6]).

Let us consider an annulus containing the unit circle $\mathbb{T} = \{z\colon |z| = 1\}$. For $\rho > 1$, let us denote by $A_\rho = \{z\colon 1/\rho < |z| < \rho\}$ and by $H(A_\rho)$ the set of holomorphic functions in A_ρ.

In this situation we can define a differential operator which is closely connected to the Sobolev inner product.

Theorem 4. *The differential operator $\mathscr{L} : H(A_\rho) \to H(A_\rho)$ defined by*

$$\mathscr{L}(y(z)) = \frac{1}{\lambda}(z^2 y''(z) + z y'(z)) + G(z) y(z), \quad y \in H(A_\rho)$$

has the following properties:

(i) $\forall f, g \in H(A_\rho)\ \langle f, g\rangle_s = \langle \mathscr{L}(f), g\rangle_\theta$.
(ii) $\mathscr{L}$ *is injective.*

Proof. (i) It suffices to prove that

$$\langle z^n, z^m\rangle_s = \langle \mathscr{L}(z^n), z^m\rangle_\theta, \quad \forall n,\ m \in \mathbb{Z}.$$

Indeed,

$$\langle \mathscr{L}(z^n), z^m\rangle_\theta = \int_0^{2\pi} \mathscr{L}(z^n)\bar{z}^m \frac{d\theta}{2\pi} = \int_0^{2\pi} \left(\frac{n^2}{\lambda} + G(z)\right) z^n \bar{z}^m \frac{d\theta}{2\pi}$$

$$= \frac{n^2}{\lambda}\int_0^{2\pi} z^n \bar{z}^m \frac{d\theta}{2\pi} + \int_0^{2\pi} z^n \bar{z}^m G(z)\frac{d\theta}{2\pi} = \frac{n^2}{\lambda}\delta_{n-m,0} + c_{n-m} = \langle z^n, z^m\rangle_s.$$

(ii) If there exists $f \in H(A_\rho)$ such that $\mathscr{L}(f) = 0$, then $\|f\|_s = 0$. Therefore $f = 0$. □

Theorem 5. *Let $f \in H(A_\rho)$ such that $f = \sum_{n=0}^{\infty} a_n \Phi_n$. Then $\mathscr{L}(f) = \sum_{k=-h}^{\infty} \hat{g}(k) z^k \in H(A_\rho)$ satisfies*

$$\forall n \geqslant 0, \quad a_n \|\Phi_n\|_s^2 = \sum_{k=0}^{n} \hat{g}(k)\overline{A_{n,k}}, \tag{13}$$

$$\text{for } 0 < k \leqslant h, \quad \hat{g}(-k) = \sum_{n=0}^{\infty} a_n \left(\sum_{i=0}^{n} A_{n,i} c_{i+k}\right). \tag{14}$$

Proof. Let $f \in H(A_\rho)$ such that $f = \sum_{n=0}^{\infty} a_n \Phi_n$. Since $\langle f, \Phi_n \rangle_s = \langle \mathscr{L}(f), \Phi_n \rangle_\theta$, we have

$$a_n \|\Phi_n\|_s^2 = \left\langle \sum_{k=-h}^{\infty} \hat{g}(k) z^k, \Phi_n \right\rangle_\theta = \left\langle \sum_{k=-h}^{\infty} \hat{g}(k) z^k, \sum_{k=0}^{n} A_{n,\,k} z^k \right\rangle_\theta = \sum_{k=0}^{n} \hat{g}(k) \overline{A_{n,\,k}}.$$

On the other hand, for $0 < k \leqslant h$, $\langle f, z^{-k} \rangle_s = \langle \mathscr{L}(f), z^{-k} \rangle_\theta = \hat{g}(-k)$, that is

$$\begin{aligned} \hat{g}(-k) &= \langle f, z^{-k} \rangle_\mu = \left\langle \sum_{n=0}^{\infty} a_n \Phi_n, z^{-k} \right\rangle_\mu \\ &= \sum_{n=0}^{\infty} a_n \left\langle \sum_{i=0}^{n} A_{n,\,i} z^i, z^{-k} \right\rangle_\mu = \sum_{n=0}^{\infty} a_n \left(\sum_{i=0}^{n} A_{n,\,i} c_{i+k} \right). \qquad \square \end{aligned}$$

Corollary 3. *For each n, the monic Sobolev orthogonal polynomial Φ_n satisfies the following second order differential equation*:

$$\mathscr{L}(y) = g_n$$

with

$$g_n(z) = \sum_{k=1}^{h} \hat{g}(-k) z^{-k} + \sum_{k=0}^{h} \hat{g}(n+k) z^{n+k}$$

and the coefficients are given by

$$\begin{pmatrix} \hat{g}(-1) \\ \vdots \\ \hat{g}(-h) \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \cdots & c_h \\ \vdots & \vdots & & \vdots \\ c_{h-1} & c_h & \cdots & 0 \\ c_h & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} A_{n,0} \\ \vdots \\ A_{n,h-1} \end{pmatrix},$$

$$\hat{g}(n) = \|\Phi_n\|_s^2,$$

$$\begin{pmatrix} \hat{g}(n+1) \\ \vdots \\ \hat{g}(n+h) \end{pmatrix} = \begin{pmatrix} \overline{c_1} & \overline{c_2} & \cdots & \overline{c_h} \\ \vdots & \vdots & & \vdots \\ \overline{c_{h-1}} & \overline{c_h} & \cdots & 0 \\ \overline{c_h} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} A_{n,n-1} \\ \vdots \\ A_{n,n-h+1} \end{pmatrix}.$$

Proof. From the preceding Theorem 5, it is immediate to obtain $\hat{g}(-k)$ for $k = 1, \ldots, h$. For the remaining coefficients, it is more clear to obtain them directly as follows: For $k = 0$, $\|\Phi_n\|_s^2 = \langle g_n, z^n \rangle_\theta = \hat{g}(n)$. For $k = 1, \ldots, h$, $\langle \Phi_n, z^{n+k} \rangle_\mu = \langle \Phi_n, z^{n+k} \rangle_s = \langle g_n, z^{n+k} \rangle_\theta = \hat{g}(n+k)$. $\square$

Remark 1. For the particular cases, $d\mu(\theta) = \frac{d\theta}{2\pi}$, the differential equation satisfied by $\Phi_n(z) = z^n$ is given by $\mathscr{L}(y) = (1 + n^2/\lambda)z^n$, and for $d\mu(\theta) = |B_1(e^{i\theta})|^2 d\theta/2\pi$ the differential equation satisfied by $\Phi_n(z)$ is $\mathscr{L}(y) = \Phi_n(0)c_1(1/z) + (\|\Phi_n\|_s^2 + \overline{c_1}z)z^n$.

Conversely, we can also obtain the following result.

Corollary 4. *Let us consider the differential equation* $\mathscr{L}(y) = g_n$, *with*

$$g_n(z) = \sum_{k=1}^{h} \hat{g}(-k)z^{-k} + z^n \sum_{k=0}^{h} \hat{g}(n+k)z^k$$

and $\hat{g}(n) \neq 0$.

If a monic polynomial of degree n, P_n, *satisfies* $\mathscr{L}(P_n) = g_n$, *then the coefficients must be given by* $\hat{g}(i) = \langle P_n, z^i \rangle_s$, *and* $P_n = \Phi_n$.

Now we can solve the following problem: Given $g \in H(A_\rho)$, find f holomorphic in a disk containing the unit circle $\mathbb{T}$ such that $\mathscr{L}(f) = g$ in an annulus containing $\mathbb{T}$.

Theorem 6. *Let* $g \in H(A_\rho)$ *such that* $g(z) = g_-(z) + g_+(z) = \sum_{k=1}^{h} \hat{g}(-k)z^{-k} + \sum_{k=0}^{\infty} \hat{g}(k)z^k$ *and let us consider the differential equation* $\mathscr{L}(y(z)) = g(z)$ *for* $z \in A_\rho$. *Then there exists* $f \in H(D_r)$ *with* $r > 1$, $f = \sum_{n=0}^{\infty} a_n \Phi_n$ *such that* $\mathscr{L}(f) = g$ *in* $D_r \cap A_\rho$ *if and only if relations* (13) and (14) *hold.*

Proof. ($\Rightarrow$) This is a consequence of Theorem 5.

($\Leftarrow$) Let $f = \sum_{n=0}^{\infty} a_n \Phi_n$ with a_n satisfying (13). Taking into account (4), (9), and (10), we obtain for $n \geqslant N$

$$|a_n| = \frac{|\sum_{k=0}^{n} \hat{g}(k)\overline{A_{n,k}}|}{\|\Phi_n\|_s^2} \leqslant \frac{\lambda}{n^2} \sum_{k=0}^{n} |\hat{g}(k)||A_{n,k}|.$$

Since $g \in H(A_\rho)$, we have $\overline{\lim}_{k\to\infty} \sqrt[k]{|\hat{g}(k)|} \leqslant 1/\rho$. Let r be such that $\overline{\lim}_{k\to\infty} \sqrt[k]{|\hat{g}(k)|} < 1/r < 1$. Then there exists N such that $\forall n \geqslant N$, $|\hat{g}(n)| < (1/r)^n$. If we take $\delta = 1/r$, we have, for $n \geqslant N$,

$$|a_n| \leqslant \frac{\lambda}{n^2} \left(\sum_{k=0}^{N-1} |\hat{g}(k)||A_{n,k}| + \sum_{k=N}^{n} |\hat{g}(k)||A_{n,k}| \right)$$

$$\leqslant \frac{\lambda}{n^2} \left(\sum_{k=0}^{N-1} H \left(\frac{1}{r}\right)^{n-k-h} M + \sum_{k=N}^{n} \left(\frac{1}{r}\right)^{k} \left(\frac{1}{r}\right)^{n-k-h} M_1 \right)$$

with $H=\max\{|\hat{g}(k)|: k=0,\dots,N-1\}$ and $M_1=M/N^2$. Therefore,

$$|a_n| \leqslant \frac{\lambda}{n^2} M_2 \left(\sum_{k=0}^{N-1} Hr^k \left(\frac{1}{r}\right)^{n-h} + \sum_{k=N}^{n} \left(\frac{1}{r}\right)^{n-h} \right)$$

$$\leqslant \frac{\lambda}{n^2} M_2 \left(\sum_{k=0}^{N-1} Hr^{N-1} \left(\frac{1}{r}\right)^{n-h} + \sum_{k=N}^{n} \left(\frac{1}{r}\right)^{n-h} \right)$$

with $M_2=\max\{M,M_1\}$.

Thus $|a_n| \leqslant (\lambda/n^2)M_3 2n(1/r)^{n-h} = (P/n)(1/r)^{n-h}$, with $M_3 = M_2\max\{Hr^{N-1},1\}$.

Therefore, $f=\sum_{n=0}^{\infty} a_n\Phi_n = \sum_{n=0}^{\infty} f_n z^n$, with $f_k=\sum_{n=k}^{\infty} a_n A_{n,k}$ is such that for $k \geqslant N$

$$|f_k| = \left|\sum_{n=k}^{\infty} a_n A_{n,k}\right| \leqslant \sum_{n=k}^{\infty} |a_n|\,|A_{n,k}| \leqslant \sum_{n=k}^{\infty} \frac{P}{n}\left(\frac{1}{r}\right)^{n-h}\left(\frac{1}{r}\right)^{n-k-h} M_1$$

$$= PM_1 r^{2h+k} \sum_{n=k}^{\infty}\left(\frac{1}{r}\right)^{2n}\frac{1}{n} \leqslant \frac{PM_1}{k} r^{2h+k} \sum_{n=k}^{\infty}\left(\frac{1}{r}\right)^{2n} = \frac{PM_1 r^{2h-k}}{k(1-1/r^2)}.$$

Thus $\overline{\lim}_{k\to\infty} \sqrt[k]{|f_k|} \leqslant 1/r$ and hence $f\in H(D_r)$, and from (14) it follows that $\mathscr{L}(f)=g$. □

Next, we solve a Dirichlet boundary value problem. First we introduce the Hardy–Sobolev spaces, which are more convenient for our purposes. Let us consider the Hardy space

$$H_2(D) = \left\{ f\in H(D);\, f(z)=\sum_{n=0}^{\infty} a_n z^n,\ \text{for } z\in D \text{ such that } \sum_{n=0}^{\infty}|a_n|^2 < +\infty \right\},$$

which is isometric to the subspace L_2^+ of L_2:

$$L_2^+ = \{g\in L_2:\ \hat{g}(-n)=0 \text{ for } n=1,2,\dots\}$$

and $\|f\|_\theta = (\sum_{n=0}^{\infty}|a_n|^2)^{1/2}$ (see [5]).

We can define the Hardy–Sobolev space

$$\mathrm{HS}_2(D) = \{f\in H_2(D):\ f'\in H_2(D)\},$$

with norm $\|f\|_{s_\theta}^2 = \|f\|_\theta^2 + \|f'\|_\theta^2$. The space $\mathrm{HS}_2(D)$ is a Hilbert space with reproducing kernel $K(z,y)=\sum_{n=0}^{\infty} z^n\bar{y}^n/(1+n^2)$.

Now, let us assume that in our Sobolev inner product, the measure μ is $\mathrm{d}\mu(\theta)=|B_h(\mathrm{e}^{\mathrm{i}\theta})|^2(\mathrm{d}\theta/2\pi)$ and $B_h(\mathrm{e}^{\mathrm{i}\theta})\neq 0$, $\forall\theta\in[0,2\pi]$. In this situation we have two equivalent norms in L_2 ($\|\cdot\|_\theta \sim \|\cdot\|_\mu$). Therefore, we can consider the following norms in $\mathrm{HS}_2(D)$, $\|\cdot\|_{s_\theta}$ and $\|\cdot\|_s$, which are also equivalent.

The differential operator $\mathscr{L}$ can be considered from $\mathrm{HS}_2(D)$ as follows $\mathscr{L}$: $\mathrm{HS}_2(D) \to H(D^*)$, with $D^*=D-\{0\}$.

Theorem 7. *Let us consider the differential equation* $\mathscr{L}(y(z)) = g(z)$ *for* $z \in \mathbb{T}$ *and let* g *be*

$$g(z) = \sum_{k=1}^{h} \hat{g}(-k)z^{-k} + \sum_{k=0}^{\infty} \hat{g}(k)z^{k} \in L_2.$$

Then there exists $f \in H(D)$ *such that* $\mathscr{L}(f) = g$ *a.e.* for $z \in \mathbb{T}$ *if and only if relations* (13) *and* (14) *hold.*

Furthermore, in this case, the differential equation holds in D^*.

Proof. ($\Rightarrow$) This is a consequence of Theorem 5.

($\Leftarrow$) We assume that (13) and (14) hold and define $f = \sum_{k=0}^{\infty} a_k \Phi_k$ with $a_k = \langle g, \Phi_k\rangle_\theta / \|\Phi_k\|_s^2$. Then $\mathscr{L}(f) \in H(D^*)$, and $|a_k| \leqslant \|g\|_\theta \|\Phi_k\|_\theta / \|\Phi_k\|_s^2 \leqslant C/\|\Phi_k\|_s^2$, and

$$\|f\|_s^2 = \sum_{k=0}^{\infty} |a_k|^2 \|\Phi_k\|_s^2 \leqslant \sum_{k=0}^{\infty} \frac{C^2}{\|\Phi_k\|_s^2} < +\infty,$$

which implies $f \in \mathrm{HS}_2(D)$. Thus f and f' are defined a.e. in $\mathbb{T}$ and f'' is defined in D.

Finally,

$$\langle \mathscr{L}(f) - g, \Phi_k\rangle_\theta = \langle \mathscr{L}(f), \Phi_k\rangle_\theta - \langle g, \Phi_k\rangle_\theta = 0$$

and therefore

$$\langle \mathscr{L}(f) - g, z^n\rangle_\theta = \langle \mathscr{L}(f) - g, \Phi_n + \alpha_{n,n-1}\Phi_{n-1} + \cdots + \alpha_{n,n-h}\Phi_{n-h}\rangle_\theta = 0.$$

Hence $\mathscr{L}(f) = g$ in D^*, that is

$$\frac{1}{\lambda} z^2 f''(z) = g(z) - \frac{1}{\lambda} z f'(z) - G(z) f(z)$$

and so $z^2 f''(z)$ is defined a.e. in $\mathbb{T}$, and $\mathscr{L}(f) = g$ a.e. in $\mathbb{T}$. $\square$

Next, we are going to present an integral representation for the solution of this last problem (see [3]).

Remark 2. If we denote by $K_s(z, y) = \sum_{n=0}^{\infty} \Phi_n(z)\overline{\Phi_n(y)}/\|\Phi_n\|_s^2$, then it is easy to verify the following properties:

(i) $K_s(z, y)$ is defined in $\bar{D} \times \bar{D}$ and it is absolutely convergent.

(ii) In the space $\mathrm{HS}_2(D)$, the function $K_s(z, y)$ is also a reproducing kernel:

$$\langle K_s(z, y), f(z)\rangle_s = \overline{f(y)}.$$

Corollary 5. *The solution* f *of the differential equation* $\mathscr{L}(y) = g$ *considered in Theorem* 7 *can be computed by*

$$f(z) = \langle g(y), K_s(y, z)\rangle_\theta.$$

Moreover by using an approximation of the kernel $K_s(z, y)$, *we can obtain an approximation of the solution.*

Proof. The first part is immediate from Theorem 7 and the preceding remark. For the second part, given $\varepsilon' > 0$, we can take m such that if we define

$$\tilde{K}(z,y) = \sum_{n=0}^{m} \frac{\Phi_n(z)\overline{\Phi_n(y)}}{\|\Phi_n\|_s^2} + \sum_{n=m+1}^{\infty} \frac{z^n \bar{y}^{\,n}}{1+n^2/\lambda},$$

then we have that for $0 < \delta \leqslant |z|, |y| \leqslant 1$,

$$|K_s(z,y) - \tilde{K}(z,y)| < \varepsilon' \quad \text{for } \delta \leqslant |z|, |y| \leqslant 1.$$

Indeed,

$$\begin{aligned} |K_s(z,y) - \tilde{K}(z,y)| &= \left| \sum_{n=m+1}^{\infty} \frac{\Phi_n(z)\overline{\Phi_n(y)}}{\|\Phi_n\|_s^2} - \sum_{n=m+1}^{\infty} \frac{z^n \bar{y}^{\,n}}{1+n^2/\lambda} \right| \\ &\leqslant \sum_{n=m+1}^{\infty} |z^n \bar{y}^{\,n}| \frac{|(\Phi_n(z)/z^n)(\overline{\Phi_n(y)}/\bar{y}^{\,n})|}{\|\Phi_n\|_s^2} + \sum_{n=m+1}^{\infty} \frac{1}{1+n^2/\lambda} \\ &\leqslant \sum_{n=m+1}^{\infty} \frac{\lambda}{n^2} \left| \frac{\Phi_n(z)}{z^n} \frac{\overline{\Phi_n(y)}}{\bar{y}^{\,n}} \right| + \sum_{n=m+1}^{\infty} \frac{1}{1+n^2/\lambda}. \end{aligned}$$

Now, if we apply Theorem 3: $|\Phi_n(z)/z^n| < 1 + \varepsilon''$ uniformly for $|z| > \delta$, we find

$$|K_s(z,y) - \tilde{K}(z,y)| < (1+\varepsilon'')^2 \sum_{n=m+1}^{\infty} \frac{\lambda}{n^2} + \sum_{n=m+1}^{\infty} \frac{1}{1+n^2/\lambda} < \varepsilon'.$$

Therefore for $\delta < |z| \leqslant 1$

$$\left| \int_0^{2\pi} (K_s(z,y) - \tilde{K}(z,y)) g(y) \frac{d\theta}{2\pi} \right| \leqslant \int_0^{2\pi} |(K_s(z,y) - \tilde{K}(z,y)) g(y)| \frac{d\theta}{2\pi} < \varepsilon' \|g\|_\theta < \varepsilon,$$

that is

$$\left| f(z) - \int_0^{2\pi} \tilde{K}(z,y) g(y) \frac{d\theta}{2\pi} \right| < \varepsilon \quad \text{for } \delta < |z| \leqslant 1. \qquad \square$$

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